

SPRINGER REPRESENTATIONS ON THE KHOVANOV SPRINGER VARIETIES

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ABSTRACT. Springer varieties are studied because their cohomology carries a natural action of the symmetric group S_n and their top-dimensional cohomology is irreducible. In his work on tangle invariants, Khovanov constructed a family of Springer varieties X_n as subvarieties of the product of spheres $(S^2)^n$. We show that if X_n is embedded antipodally in $(S^2)^n$ then the natural S_n -action on $(S^2)^n$ induces an S_n -representation on the image of $H_*(X_n)$. This representation is the Springer representation. Our construction admits an elementary (and geometrically natural) combinatorial description, which we use to prove that the Springer representation on $H_*(X_n)$ is irreducible in each degree. We explicitly identify the Kazhdan-Lusztig basis for the irreducible representation of S_n corresponding to the partition $(n/2, n/2)$.

1. INTRODUCTION

Springer varieties are a collection of varieties that arise in two mathematical areas: geometric representation theory and low-dimensional topology. Springer varieties were originally investigated because their cohomology carries a graded representation of the symmetric group S_n and their top-dimensional cohomology is irreducible. Many geometric constructions of the Springer representation exist; [3] has a survey. One thing geometric constructions share is that explicit calculations are impossible outside of special cases.¹

In important recent work [9, 10] Khovanov studied a family of rings related to an invariant of tangle cobordisms. He discovered the center of each ring was isomorphic to the cohomology of the Springer variety corresponding to the partition $(n/2, n/2)$, which we call X_n . He conjectured that X_n could be realized as a subspace of the n -fold product of spheres $(S^2)^n$, with components homeomorphic to $(S^2)^{n/2}$ and indexed by noncrossing matchings; this follows from recent work of Cautis-Kamnitzer [2] as shown in the Appendix of this paper. Permutation representations arise in the work of Khovanov et alia, but for the group $S_{n/2}$ rather than S_n .

This paper brings together the two themes of geometric representation theory and low-dimensional topology. We obtain an explicit geometric and combinatorial construction of the Springer action for X_n . Unlike other presentations of Springer representations, our construction is clear and elementary—both geometrically and combinatorially. We use this explicit construction to identify the Springer representations in *every* degree of homology and to identify the Kazhdan-Lusztig basis explicitly and simply in top-degree homology.

¹Using an algebraic combinatorial construction of the cohomology ring, Garsia-Procesi identified the Springer representation in all degrees.

The variety $(S^2)^n$ and its homology have a natural S_n -action given by permuting the copies of S^2 . We use an antipodal map $\gamma : X_n \rightarrow (S^2)^n$ to embed X_n in the product of spheres and prove that S_n restricts to a permutation action on the image $\gamma_*(H_*(X_n))$. Not every geometric map $\alpha : X_n \rightarrow (S^2)^n$ satisfies this property; indeed if α is the identity embedding then $\alpha_*(H_*(X_n))$ is in general not an S_n -subrepresentation of $H_*((S^2)^n)$.

We prove four main theorems in this paper:

- that there is a well-defined S_n -action on $\gamma_*(H_*(X_n))$ (Theorem 3.14);
- that the geometric representation is described combinatorially as an action of S_n on noncrossing matchings that permutes the strands of the matchings (Theorem 3.14);
- that the S_n -action on $\gamma_*(H_*(X_n))$ is the Springer representation (Theorem 4.3);
- and that the Springer representation on $H_k(X_n)$ is the irreducible representation corresponding to the partition $(n - k, k)$ (Theorem 4.3).

A search of the literature shows that “the Springer representation” can mean any of: Springer’s original representation on the cohomology of Springer varieties [15], any other construction of the same representation, a construction of Springer’s representation tensored with the sign representation [11, 13, 1], or a homology representation that is isomorphic to Springer’s cohomology representation [8]. (Hotta proves many of these equivalences in [7].)

Our construction gives an S_n -representation on homology with complex coefficients. We prove it is the Springer representation by comparing it explicitly with the de Concini-Procesi construction of Springer representations [4], using tools from Garsia-Procesi’s combinatorial analysis of de Concini-Procesi’s work [6]. Our description of the homology of the Springer variety X_n uses techniques from skein modules developed by Russell in [12], unlike de Concini-Procesi’s presentation, which relies on Borel’s quotient presentation of the cohomology of the flag variety.

We exploit two different antipodal maps $X_n \rightarrow (S^2)^n$ in Section 3.1. The first is a geometric embedding of the entire variety, defined to be the identity on even coordinates and the antipodal map on odd coordinates. The second is an embedding of each component of X_n ; it is more natural in topological contexts and useful for computations but does not extend to all of X_n . Corollary 3.7 proves the image $\gamma_*(H_*(X_n))$ is the span of the images of antipodal maps on each component, permitting us to use both types of antipodal maps in our calculations.

The product basis of $H_*((S^2)^n)$ allows us to identify the S_n -representation on $\gamma_*(H_*(X_n))$ with a subspace of Young tabloids, a classical construction of S_n -representations. The tabloid sums corresponding to basis elements in $\gamma_*(H_*(X_n))$ are not the classical basis for Specht modules; our methods allow us to conclude that we obtain the Kazhdan-Lusztig basis in top degree, and we conjecture that this is true in all degrees. We show that for each k the appropriate Specht module intersects $\gamma_*(H_k(X_n))$ nontrivially. The dimensions of the two subspaces agree, so the image $\gamma_*(H_k(X_n))$ is irreducible and the S_n -representation we defined on $H_*(X_n)$ is the Springer representation.

We address this paper to several audiences: geometric representation theorists, low-dimensional topologists, and classical representation theorists. Consequently we include material that each audience may find remedial.

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2. THE SPRINGER FIBER X_n

2.1. Classical results about the Springer action on $H_*(X_n)$. Springer varieties are subvarieties of the flag variety, which is the collection of nested subspaces $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{n-1} \subseteq \mathbb{C}^n$ so that each V_i is an i -dimensional subspace of \mathbb{C}^n . A *nilpotent* linear operator $X : \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a linear operator whose only eigenvalue is zero. The Springer variety associated to a nilpotent linear operator X is the collection of flags fixed by X , namely

$$\{\text{Flags } V_\bullet : XV_i \subseteq V_i \text{ for all } i\}.$$

If two linear operators are conjugate then the corresponding Springer varieties are homeomorphic. The conjugacy class corresponding to X is determined by the partition of n given by the Jordan blocks of X . We only study representatives of the homeomorphism class of a Springer variety, so for our purposes Springer varieties are indexed by partitions.

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (\mu_1, \dots, \mu_n)$ be partitions of n written in decreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n \geq 0$ with some parts possibly zero. We often omit terminal zeros in our notation. We also interchangeably use the vocabulary for partitions and for Young diagrams (collections of left- and top-aligned boxes so that the i^{th} row has λ_i boxes for each i); hence λ_i can be called either the i^{th} part of the partition or the i^{th} row of the Young diagram. The Springer variety corresponding to λ is denoted X_λ . The irreducible representation corresponding to λ is discussed in a later section; we often call the representation λ as well.

We use three facts about the Springer representation. The first is a classical result of Springer theory.

Proposition 2.1. (Springer) *The S_n -module $H^{\text{top}}(X_\lambda)$ is the irreducible representation corresponding to λ .*

Moreover the dimension of X_λ is determined in [14] (among others) to be

$$(1) \quad \sum_{i=1}^n (i-1)\lambda_i.$$

The second was proven by De Concini-Procesi in [4, page 213]:

Proposition 2.2. (De Concini-Procesi) *The multiplicity of the irreducible representation μ in the ungraded representation $H^*(X_\lambda)$ is the number of fillings of μ with the integers $\{1^{\lambda_1}, 2^{\lambda_2}, \dots, n^{\lambda_n}\}$ so that rows increase weakly and columns increase strictly.*

Thus we deduce the following lemma.

Lemma 2.3. *If $\lambda = (n - j, j)$ is a two-row partition then the multiplicity of the irreducible representation μ in $H^*(X_\lambda)$ is either zero or one. It is one if μ is a two-row partition of n with $\mu_1 \geq \lambda_1$; it is zero otherwise.*

Proof. The Young diagram μ cannot be filled with $\{1^{\lambda_1}, 2^{\lambda_2}\}$ and have strictly-increasing columns unless μ has at most two rows and the second row has size at most $\mu_2 \leq \lambda_2$. In this case there is exactly one filling with strictly-increasing columns and weakly-increasing rows, namely the filling with $1^{\lambda_1}, 2^{\lambda_2 - \mu_2}$ on the top row and 2^{μ_2} on the second row. \square

The third fact was proven by Garsia-Procesi in [6, Equation 4.2]:

Proposition 2.4. (Garsia-Procesi) *Suppose that λ and μ are partitions so that for each i the sums $\mu_1 + \mu_2 + \cdots + \mu_i \geq \lambda_1 + \lambda_2 + \cdots + \lambda_i$. Then there is a graded S_n -equivariant surjection $H^*(X_\lambda) \twoheadrightarrow H^*(X_\mu)$.*

From this together with the previous we conclude:

Proposition 2.5. *Let λ be any partition of n with at most two rows. Then $H^k(X_\lambda)$ is the irreducible representation corresponding to the partition $(n - k, k)$ for each $k \leq \lambda_2$.*

Proof. The proof is by induction on λ_2 . If $\lambda_2 = 0$ then the Springer variety X_λ consists of a single flag, and $H^k(X_\lambda)$ is the trivial representation when $k = 0$ and is zero otherwise. This proves the base case.

Now assume the claim holds for the partition $\lambda' = (n - j + 1, j - 1)$ and let $\lambda = (n - j, j)$. The variety X_λ has dimension λ_2 by the formula in Equation (1). The top-dimensional cohomology $H^{\lambda_2}(X_\lambda)$ is the irreducible representation corresponding to λ by the first property of Springer varieties. Let $m_{\mu, \lambda}$ denote the multiplicity of the irreducible representation μ in

$H^*(X_\lambda)$ and similarly for $m_{\mu, \lambda'}$. The cohomology surjection $H^k(X_\lambda) \twoheadrightarrow H^k(X_{\lambda'})$ is an S_n -equivariant map for each k . Hence $m_{\mu, \lambda} \geq m_{\mu, \lambda'}$ for each irreducible μ . Lemma 2.3 proved that $m_{\mu, \lambda} = m_{\mu, \lambda'}$ for each μ with $\mu_2 \leq \lambda'_2$, so the cohomology surjection must be an isomorphism when $k < j$. Induction proves the claim. \square

The number of standard fillings of $(n - k, k)$ is the rank of the irreducible representation corresponding to the partition $(n - k, k)$. Hence we obtain a corollary.

Corollary 2.6. *If λ is a two-row partition $(n - j, j)$ and $k \leq j$ then the rank of $H^k(X_\lambda)$ is the number of standard fillings of $(n - k, k)$, namely fillings of $(n - k, k)$ with $\{1, 2, \dots, n\}$ without repetition so that both rows and columns increase.*

Most of the classical theory of Springer representations is written for cohomology rather than homology. The next lemma is included for the sake of completeness.

Lemma 2.7. *If k satisfies $0 \leq k \leq n$ then over complex coefficients $H^k(X_n) \cong H_k(X_n)$.*

Proof. From Corollary 2.6 we know $H^k(X_n) \cong \mathbb{C}^{b_k}$ where b_k is the number of standard fillings of $(n - k, k)$. Say that $H_k(X_n) \cong \mathbb{C}^{a_k}$ for some nonnegative integer a_k . By definition $H^k(X_n) = \text{Hom}_{\mathbb{C}}(H_k(X_n), \mathbb{C})$. Since $\text{Hom}_{\mathbb{C}}(\mathbb{C}^{a_k}, \mathbb{C}) \cong (\text{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}))^{a_k} \cong \mathbb{C}^{a_k}$ we conclude $a_k = b_k$ for all k . \square

2.2. The Khovanov construction. Let $n \in \mathbb{N}$ be even. In [9] Khovanov proves for $\lambda = (n/2, n/2)$ the cohomology ring $H^*(X_\lambda)$ is isomorphic to the center of a ring related to his tangle invariant by proving both are isomorphic to the cohomology ring of a certain subspace

\tilde{S} of $(S^2)^n$. Results from recent work of Cautis-Kamnitzer [2] imply that X_λ and \tilde{S} are homeomorphic; the Appendix has a complete proof. This section reviews the construction

of \tilde{S} , which we call the Khovanov construction of the $(n/2, n/2)$ Springer variety, and gives

a diagrammatic set of generators for $H_*(\tilde{S})$.

Definition 2.8. *A noncrossing matching on n vertices is a collection of $n/2$ disjoint arcs which connect n vertices (on a straight line) pairwise.*

Let $B^{n/2}$ be the set of all noncrossing matchings on n vertices. Given $a \in B^{n/2}$, let $S_a = \{(x_1, \dots, x_n) \in (S^2)^n : x_i = x_j \text{ if } (i, j) \in a\}$. Khovanov's construction of the $(n/2, n/2)$

Springer variety, denoted \tilde{S} , is the union of S_a over all matchings

$$\tilde{S} = \bigcup_{a \in B^{n/2}} S_a$$

Generators for the homology $H_*(\tilde{S})$ can be described completely combinatorially.

Definition 2.9. *A dotted noncrossing matching is a noncrossing matching in which some arcs have a single dot each. We define the dotted arcs of the matching to be those arcs with dots on them.*

We will use a, b to denote noncrossing matchings and M, M' to denote dotted noncrossing matchings. There are many possible dotted noncrossing matchings associated to each noncrossing matching. Figure 2.1 gives an example. Dotted noncrossing matchings are a



FIGURE 2.1. A dotted noncrossing matching (for the generator $[p \times c \times c]$)

simple, diagrammatic way to represent the homology generators corresponding to cells in the cartesian product CW-decomposition for S_a . This was first described in [12, Section 5.2].

Proposition 2.10. (Russell) *The different dotted noncrossing matchings for the noncrossing matching a are naturally bijective with the homology generators for $H_*(S_a)$.*

Proof. The standard CW-decomposition of the two-dimensional sphere S^2 is given by a fixed point $p \in S^2$ and the two-cell $c = S^2 - p$. A cartesian product of spheres is endowed with

the cartesian product cell decomposition and homology basis. Since S_a is diffeomorphic to $(S^2)^{n/2}$ we may fix such a cell decomposition. Each arc in a matching a represents one sphere in the cartesian product S_a . Each generator of $H_*(S_a)$ consists of the choice of a point or a two-cell for each of these spheres. Given a dotted noncrossing matching M associated to a , the corresponding homology generator chooses a point for the homology generator of a sphere if and only if the corresponding arc in M has a dot. \square

Because of this proposition we will often refer to the dotted noncrossing matchings and the corresponding homology generators interchangeably. Figure 2.1 shows an example of a generator for $H_*(S_a)$ where a is the matching on 6 vertices.

Varying over all possible markings of a single matching yields a generating set for $H_*(S_a)$. By considering all possible markings of all noncrossing matchings on n vertices, we get a set of generators for $\bigoplus_a H_*(S_a)$.

In [12, Section 3.2] Russell defines Type I and Type II relations on dotted noncrossing matchings.

Definition 2.11. *Let $a, b \in B^{n/2}$ with all arcs in a and b identical except $(i, j), (k, l) \in a$ and $(i, l), (j, k) \in b$ for some $i < j < k < l$. Fix a choice of dots on the arcs of a in positions other than i, j, k, l .*

- (1) **Type I relations:** *Let M_1 be the dotted noncrossing matching with dotted arc (i, j) and undotted arc (k, l) . Let M_2 be the dotted noncrossing matching with dotted arc (k, l) and undotted arc (i, j) . Define M'_1 and M'_2 analogously for b , so that they agree with the M_i off of i, j, k, l . Type I relations have the form*

$$M_1 + M_2 - M'_1 - M'_2 = 0.$$

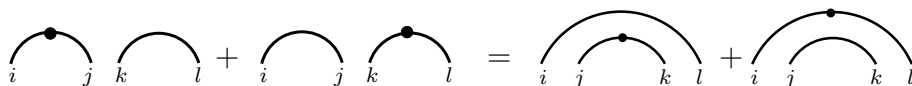


FIGURE 2.2. Type I relation

- (2) **Type II relations:** *Let M_3 be the dotted noncrossing matching with dotted arcs (i, j) and (k, l) . Define M'_3 analogously for b . Type II relations have the form*

$$M_3 - M'_3 = 0.$$

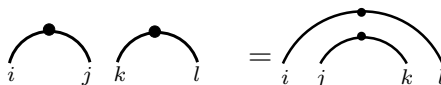


FIGURE 2.3. Type II relation

Appealing to Type I and Type II relations [12, Theorem 3.1] proved the following.

Proposition 2.12. (Russell) *Let R be the subspace of $\bigoplus_a H_*(S_a)$ generated by all Type I and Type II relations. Then*

$$H_*(\tilde{S}) \cong \left(\bigoplus_a H_*(S_a) \right) / R.$$

Each undotted arc corresponds to a two-cell and each dotted arc corresponds to a point in the cartesian product CW-decomposition of S_a , which describes the underlying topology of the relations. Figures 2.2 and 2.3 show both types. Note that there is a fixed arrangement of dotted and undotted arcs compatible with the vertex labelings that is not shown.

2.3. Bijection between the homology basis for X_n and standard tableaux. The generators for homology described in Section 2.2 are not linearly independent. In this section we choose a homology basis for X_n indexed by *standard* noncrossing matchings. We then describe a bijection between the standard noncrossing matchings and the standard tableaux of shape $(n - k, k)$ for $k = 0, 1, \dots, n/2$.

Definition 2.13. *A standard noncrossing matching is a (dotted) noncrossing matching in which no dotted arc is nested below another arc.*

As a consequence, the only dotted arcs in a standard noncrossing matching are either entirely unnested or the topmost arc in a nesting. Figures 2.4 and 2.5 show several examples.



FIGURE 2.4. Standard dotted noncrossing matchings

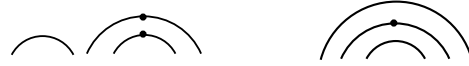


FIGURE 2.5. Nonstandard dotted noncrossing matchings

We now define a function φ from dotted noncrossing matchings to standard tableaux. Given a dotted noncrossing matching M , construct a tableau T as follows:

- For each undotted arc (i, j) with $i < j$, place the number j in the bottom row of T .
- Place all other numbers in the top row of T .
- Order each row so that the numbers increase left-to-right.

The tableau T obtained by this process is $\varphi(M)$. It is a two-row standard tableau by construction. Note that if the dotted matching M had k undotted arcs and n vertices, then the resulting tableau has shape $(n - k, k)$, as shown in Figure 2.6.

Next we define a map θ from standard two-row tableaux to standard dotted noncrossing matchings. Given a standard tableau T of shape $(n - k, k)$, construct a standard dotted noncrossing matching M as follows:

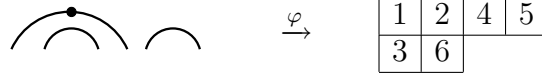
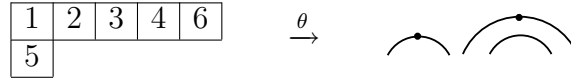


FIGURE 2.6. A homology generator and its associated tableau

- (1) Repeat the next two steps until all numbers on the second row have been matched:
 - (a) Let j be the leftmost unmatched number in the second row.
 - (b) Add an undotted arc from j to the rightmost unmatched i to the left of j in M .
(In other words $i = \max\{k : k < j, k \text{ unmatched}\}$.)
- (2) If i is the leftmost unmatched number in M , add a dotted arc from i to its nearest unmatched neighbor on the right. (In other words, add a dotted arc (i, j) where $j = \min\{k : k > i, k \text{ unmatched}\}$.)

The matching M is by definition $\theta(T)$. Figure 2.7 gives an example.

FIGURE 2.7. A tableau of shape $(5, 1)$ and its associated matching

Lemma 2.14. *For each standard two-row tableau T , the matching $M = \theta(T)$ is a standard dotted noncrossing matching. If T has shape $(n - k, k)$ then M has exactly k undotted arcs.*

Proof. Denote the column of j by $c(j)$. Each of the numbers in the first $c(j)$ positions of the first row of T are above or to the left of j . The definition of standard tableau implies that all of these numbers are less than j . Hence Step (b) can always be performed.

We confirm the matching produced by Step (1) is noncrossing. Suppose arc (i, j) was added after arc (i', j') . Step (a) ensures that j' is to the left of j in the standard tableau so $j' < j$. If the arcs cross then either $i' < i < j' < j$ or $j' < i' < i < j$. Both contradict Step (b).

There are no nested unmatched vertices after Step (1), namely unmatched vertices k with an arc (i, j) so that $i < k < j$. (Such k contradict the choice of i in Step (b).) Hence Step (2) creates neither crossing arcs nor dotted arcs that are nested by undotted arcs. (The latter would be a dotted (i, j) such that there is an undotted (i', j') with $i' < i < j < j'$.)

This shows $\theta(T)$ is a standard dotted noncrossing matching. Exactly one undotted arc was constructed for each box in the second row of T , which proves the claim. \square

Theorem 2.15. *Let n, k be a pair with n even and $k \leq n/2$. The map φ is a bijection between standard noncrossing matchings on n vertices with exactly k undotted arcs and two-row standard tableaux of shape $(n - k, k)$. Its inverse is θ .*

Proof. The composition $\varphi \circ \theta$ is the identity since the right endpoints of the undotted arcs in $\theta(T)$ are exactly the second row of T .

If M has no undotted arcs then $\varphi(M)$ is the (unique) tableau of shape (n) . The matching $\theta(\varphi(M))$ is the entirely unnested, entirely dotted matching, which has a dotted arc $(2i - 1, 2i)$ for each $i \leq n/2$. This is the only fully dotted standard noncrossing matching, since standard in this case implies that the matching has no nesting. So $\theta(\varphi(M)) = M$ when $k = 0$ (independent of n).

Assume that $\theta \circ \varphi$ is the identity on all standard noncrossing matchings with at most $k-1$ undotted arcs (for all n). Let M be a matching with k undotted arcs, let $T = \varphi(M)$, and let j be the leftmost vertex with an undotted arc (i, j) in M . Then $i = j-1$, since otherwise the arc (i, j) nests another arc (i', j') with $i < i' < j' < j$. This either violates the minimality assumption on j or the hypothesis that M is standard.

Let M^j denote the matching obtained by removing the vertices $j-1$ and j , and renaming the remaining vertices $1, \dots, n-2$. Let T^j be the standard tableau obtained by removing the entries $j-1$ and j and renaming the rest of the entries $1, \dots, n-2$. Note that $\varphi(M^j) = T^j$ and that M^j has $k-1$ undotted arcs. The inductive hypothesis says that $\theta(\varphi(M^j)) = M^j$.

We compare $\theta(T)$ to $\theta(T^j)$. The first iteration of Step (1) on T creates an undotted arc $(j-1, j)$ since j is the first entry in the second row of T and hence $j-1$ is on the first row. Successive iterations create an undotted arc (i, k) in T if and only if either $i > j$ and $(i-2, k-2)$ is an arc in T^j or $i < j-1$ and $(i, k-2)$ is an arc in T^j . Hence $\theta(T)$ is the standard matching obtained by relabeling each vertex k such that $k \geq j-1$ with the number $k+2$, and then inserting the undotted arc $(j-1, j)$ into T^j . In other words $\theta(T) = T$. This implies by induction that $\theta \circ \varphi$ is the identity on all standard noncrossing matchings. \square

Corollary 2.16. *The standard noncrossing matchings on n vertices with k undotted arcs form a basis for $H_{2k}(X_n)$.*

Proof. The noncrossing matchings on n vertices with k undotted arcs generate $H_{2k}(X_n)$, so we must show that each nonstandard noncrossing matching is equivalent to a linear combination of standard noncrossing matchings with the same number of undotted arcs. Type I relations (see Definition 2.11) permit us to replace each instance of a dotted arc nested below an undotted arc by a linear combination of matchings with the dotted arcs at the top of nestings. This means we may assume the matching is nonstandard only because dotted arcs are nested. Type II relations permit us to replace each instance of two nested dotted arcs (i, j) and (i', j') satisfying $i' < i < j < j'$ with the unnested dotted arcs (i, i') and (j, j') . Together, these relations allow us to reduce to standard form. \square

3. THE TOPOLOGICAL S_n -ACTION ON $H_*(X_n)$

The space $(S^2)^n$ has a natural action of the permutation group S_n given by permuting the components. We will use an antipodal map to embed the Springer fiber X_n into $(S^2)^n$. This gives a natural action of S_n on the image of $H_*(X_n)$ under the induced map on homology. In this section we describe this map and the S_n action on $H_*(X_n)$, giving an explicit formula for the action of each simple transposition s_i on the homology basis from Section 2.

3.1. The antipodal embedding. We will define two embeddings. The first is described as a map on the space X_n but ignores the structure coming from the individual components S_a . The second embedding, defined for each S_a , is more convenient for computational purposes but does not extend to X_n . By showing that both maps give the same representation, we can use the first as a geometric embedding of the Springer variety and the second to study the S_n -action on that embedding.

Many key calculations involve $H_*((S^2)^n)$. As in the prior sections, we use a homology basis associated to the cartesian product cell decomposition. Line diagrams, like those shown in

Figure 3.1, allow us to extend the diagrammatic notation from Section 2.2 to the cartesian product generators of $H_*((S^2)^n)$.

Definition 3.1. *A line diagram is a collection of n parallel, vertical line segments some of which are marked with a single dot.*

Each cartesian product generator for $H_*((S^2)^n)$ is represented by a dotted strand for each sphere where a point has been chosen and an undotted strand for each sphere where a two-cell has been chosen. Figure 3.1 has an example.

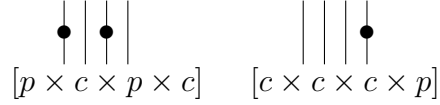


FIGURE 3.1. Generators for $H_*((S^2)^4)$

We begin building the antipodal embeddings by defining two maps f and g . There is only one noncrossing matching on two vertices, namely the arc $(1, 2)$. According to Khovanov's construction the space $X_2 = \{(x, x) : x \in S^2\}$. Define maps $f, g : X_2 \rightarrow S^2 \times S^2$ by

$$f(x, x) = (x, -x) \quad \text{and} \quad g(x, x) = (-x, x)$$

where $-x$ denotes the image of x under the antipodal map $S^2 \rightarrow S^2$. (If S^2 is realized as the solutions in \mathbb{R}^3 to $x_1^2 + x_2^2 + x_3^2 = 1$ then the antipodal map does in fact send x to $-x$ for each $x \in S^2$.)

Since $X_2 \cong S^2$ the homology $H_*(X_2)$ is generated by a point p and a two-cell c . The diagrams associated to these generators are shown in Figure 3.2.



FIGURE 3.2. Generators for $H_*(X_2)$

Lemma 3.2. *The image in homology of the maps f and g is:*

$$f_*([p]) = [p \times p] = g_*([p]) \quad \text{and} \quad f_*([c]) = [c \times p] - [p \times c] = -g_*([c]).$$

Proof. We first consider these maps applied to the zero cell p . We get $f(p) = (p, -p)$ and $g(p) = (-p, p)$. All points are homologous in a connected space so we obtain

$$[(p, p)] = [(p, -p)] = [(-p, p)] \in H_*(S^2 \times S^2).$$

Then $f_*([p]) = [p \times p] = g_*([p])$ as desired. In terms of diagrams this is

$$f_*(\text{arc with dot}) = \text{two dots} = g_*(\text{arc with dot}).$$

We now find $f_*([c])$ and $g_*([c])$. Let $\pi_1, \pi_2 : S^2 \times S^2 \rightarrow S^2$ denote projection to the respective coordinate and let $\iota_1, \iota_2 : S^2 \hookrightarrow S^2 \times S^2$ denote inclusion to the respective coordinate, for instance $\iota_1(x) = (x, p)$ for all $x \in S^2$ and a fixed $p \in S^2$.

Since $\iota_1(c) = c \times p$ we have $(\iota_1)_*([c]) = [c \times p]$ and similarly for ι_2 . In addition $\pi_1 \circ \iota_1 = id$ and $(\pi_1 \circ \iota_2)(x) = p$ for all $x \in S^2$. Hence $(\pi_1 \circ \iota_1)_*([c]) = [c]$ and $(\pi_1 \circ \iota_2)_*([c]) = 0$. We conclude that $(\pi_1)_*([c \times p]) = [c]$ and $(\pi_1)_*([p \times c]) = 0$. A symmetric argument proves $(\pi_2)_*([c \times p]) = 0$ and $(\pi_2)_*([p \times c]) = [c]$.

Define $f_0 : S^2 \rightarrow X_2$ by $f_0(x) = (x, -x)$. Note that $f_0(c) = f(c)$ and so $(f_0)_*([c]) = f_*([c])$. We compute the constants c_1, c_2 in

$$(f_0)_*([c]) = c_1[c \times p] + c_2[p \times c]$$

using the maps $\pi_1 \circ f_0$ and $\pi_2 \circ f_0$. The composition $\pi_1 \circ f_0 = id$ so $(\pi_1 \circ f_0)_*([c]) = [c]$. Applying each map sequentially gives

$$(\pi_1)_*((f_0)_*([c])) = (\pi_1)_*(c_1[c \times p] + c_2[p \times c]) = c_1[c]$$

and so $c_1 = 1$. The composition $\pi_2 \circ f_0$ is the antipodal map α and $\alpha_*([c]) = -[c]$. Hence

$$(\pi_2)_*(c_1[c \times p] + c_2[p \times c]) = c_2[c] = \alpha_*([c])$$

so $c_2 = -1$. Analogous arguments show $(g_0)_*([c]) = -(f_0)_*([c])$. In terms of diagrams this is

$$f_*(\frown) = \left| \begin{array}{c} | \\ \bullet \\ | \end{array} - \begin{array}{c} | \\ \bullet \\ | \end{array} \right| = -g_*(\frown).$$

□

For fixed (even) n and each $0 \leq m \leq \frac{n}{2}$, define $G^m : (X_2)^{\frac{n}{2}} \rightarrow (S^2)^n$ to be

$$G^m = g \times \cdots \times g \times f \times \cdots \times f$$

where g is applied to the first m copies of X_2 and f is applied to the last $n/2 - m$ copies. Let a_0 be the completely unnested matching, namely the matching with arcs $(1, 2), \dots, (n-1, n)$.

Note that $S_{a_0} = (X_2)^{n/2}$ so the maps G^m can be thought of as having domain S_{a_0} .

Lemma 3.3. *For all $0 \leq m \leq \frac{n}{2}$ and cartesian product generators $q \in H_*((X_2)^{\frac{n}{2}})$ the maps*

$$G_*^m \text{ satisfy } G_*^m(q) = \pm G_*^{n/2}(q).$$

Proof. Consider the cartesian product decomposition on $(X_2)^{n/2} = S_{a_0}$ and let $q \in H_*(S_{a_0})$ be a cartesian product generator. Then q has the form $q = [q_1 \times \cdots \times q_{n/2}]$ where each $q_i = p$

or c . Since $H_*((S^2)^n) \cong \bigotimes_{i=1}^{n/2} H_*(S^2 \times S^2)$ we have

$$\begin{aligned} G_*^m(q) &= (g \times \cdots \times g \times f \times \cdots \times f)_*(q) \\ &= g_*(q_1) \otimes \cdots \otimes g_*(q_m) \otimes f_*(q_{m+1}) \otimes \cdots \otimes f_*(q_{n/2}) \end{aligned}$$

Lemma 3.2 showed $f_*(q_i) = \pm g_*(q_i)$ for all i , and thus

$$G_*^m(q) = g_*(q_1) \otimes \cdots \otimes g_*(q_m) \otimes \pm g_*(q_{m+1}) \otimes \cdots \otimes \pm g_*(q_{n/2}) = \pm G_*^{n/2}(q).$$

□

Define the embedding $\gamma : X_n \rightarrow (S^2)^n$ as

$$\gamma(x_1, \dots, x_n) = (-x_1, x_2, \dots, -x_{n-1}, x_n) = ((-1)^i x_i)$$

where $-x$ again denotes the image of x under the antipodal map $S^2 \rightarrow S^2$. For each matching $a \in B^{n/2}$, define an embedding of the component $\gamma_a : S_a \rightarrow (S^2)^{n/2}$ to be the identity on coordinates corresponding to right endpoints of arcs and the antipodal map on coordinates corresponding to left endpoints. Explicitly let ρ_a be the binary function

$$\rho_a(i) = \begin{cases} 1 & \text{if there is } j \text{ with } (i, j) \in a \\ 0 & \text{if there is } j \text{ with } (j, i) \in a \end{cases}$$

Then $\gamma_a : S_a \rightarrow (S^2)^n$ is defined to be $\gamma_a(x_1, \dots, x_n) = ((-1)^{\rho_a(i)} x_i)$.

For example let $n = 6$ and let a be the matching in Figure 3.3. Then each $\vec{v} \in S_a$ has the



FIGURE 3.3. The matching a

form $\vec{v} = (x, y, y, z, z, x)$ where $x, y, z \in S^2$. The antipodal maps γ and γ_a applied to these elements give

$$\gamma(\vec{v}) = (-x, y, -y, z, -z, x) \quad \text{and} \quad \gamma_a(\vec{v}) = (-x, -y, y, -z, z, x).$$

The map γ_a is related to the parity of coordinates, as the next lemma shows.

Lemma 3.4. *Each arc in a noncrossing matching has one even and one odd endpoint.*

Proof. Take $a \in B^{n/2}$ and an arc $(i, j) \in a$. In order for a to be noncrossing, all vertices between i and j must be paired among themselves. This means that there are an even number of vertices between i and j , so i and j have opposite parities. □

Again let a_0 be the matching with arcs $(1, 2), (3, 4), \dots, (n-1, n)$. Let $a \in B^{n/2}$ be any matching and let m be the number of arcs in a with odd left endpoints. Order the arcs in a by listing all arcs with odd left endpoints first, and otherwise listing from least left endpoint to greatest left endpoint. This gives a list $e_1, \dots, e_{n/2}$ where the arcs e_1, \dots, e_m have odd left endpoints and the arcs $e_{m+1}, \dots, e_{n/2}$ have even left endpoints. Define σ_a to be the permutation that maps the arc $(2k-1, 2k)$ to the arc e_k so that $\sigma_a(2k-1)$ is the left endpoint of e_k and $\sigma_a(2k)$ is the right endpoint. Hence σ_a sends the matching a_0 to a . Let σ_a also denote the map that permutes coordinates of $(S^2)^n$ so that

$$\sigma_a(x_1, \dots, x_n) = (x_{\sigma_a^{-1}(1)}, \dots, x_{\sigma_a^{-1}(n)}).$$

For example consider the matching in Figure 3.3. There is only one arc with an odd left endpoint so $m = 1$. The ordered list of arcs in a is $(1, 6), (2, 3), (4, 5)$ and the permutation is $\sigma_a = (26543)$ in cycle notation. For $x, y, z \in S^2$ the map $\sigma_a : S_{a_0} \rightarrow S_a$ will be

$$(x, x, y, y, z, z) \mapsto (x, y, y, z, z, x).$$

Lemma 3.5. *Choose a matching $a \in B^{n/2}$ and let m be the number of arcs in a with odd left endpoints. Then the maps $\gamma|_{S_a}$ and $\sigma_a \circ G^m \circ \sigma_a^{-1}$ from S_a to $(S^2)^n$ are equal.*

Proof. Choose i with $1 \leq i \leq n$. The map σ_a^{-1} sends arcs in a with odd left endpoints to the arcs $(1, 2), (3, 4), \dots, (2m-1, 2m)$ and arcs with even left endpoints to $(2m+1, 2m+2), \dots, (n-1, n)$. The map G^m acts as g on the first m arcs and as f on the last $n/2 - m$ arcs; this only affects the sign of each coordinate. Then σ_a returns each coordinate to its original position.

By Lemma 3.4 each arc has one odd and one even endpoint. Hence if i is odd and a left endpoint in a then σ_a^{-1} sends i to one of $1, 3, 5, \dots, 2m-1$; if i is odd and a right endpoint in a then σ_a^{-1} sends i to one of $2m+2, 2m+4, \dots, n$. In either case

$$x_i \xrightarrow{\sigma_a^{-1}} x_{\sigma_a^{-1}(i)} \xrightarrow{G^m} -x_{\sigma_a^{-1}(i)} \xrightarrow{\sigma_a} -x_i.$$

Similarly if i is even and either a left or right endpoint in a then

$$x_i \xrightarrow{\sigma_a^{-1}} x_{\sigma_a^{-1}(i)} \xrightarrow{G^m} x_{\sigma_a^{-1}(i)} \xrightarrow{\sigma_a} x_i.$$

The composition $\sigma_a \circ G^m \circ \sigma_a^{-1}$ is the identity on even coordinates and the antipodal map on odd coordinates so $\gamma|_{S_a} = \sigma_a \circ G^m \circ \sigma_a^{-1}$. \square

Lemma 3.6. *For each matching $a \in B^{n/2}$ the maps γ_a and $\sigma_a \circ G^{n/2} \circ \sigma_a^{-1}$ from S_a to $(S^2)^n$ are equal.*

Proof. The map σ_a^{-1} sends every arc of a to an arc of the form $(2k-1, 2k)$. The map $G^{n/2}$ applies g to the coordinates corresponding to each of these arcs and σ_a returns them to their original positions. Hence the composition is the identity on coordinates corresponding to right endpoints and is the antipodal map on coordinates corresponding to left endpoints. In other words $\gamma_a = \sigma_a^{-1} \circ G^{n/2} \circ \sigma_a$. \square

Corollary 3.7. *The image $\gamma_*(H_*(X_n))$ is the vector subspace of $H_*((S^2)^n)$ generated by $\bigcup_a ((\gamma_a)_*(H_*(S_a)))$.*

Proof. Lemmas 3.5 and 3.6 proved $\gamma_a = \sigma_a \circ G^{n/2} \circ \sigma_a^{-1}$ and $\gamma|_{S_a} = \sigma_a \circ G^m \circ \sigma_a^{-1}$. Lemma 3.3 showed that $G_*^m(q) = \pm G_*^{n/2}(q)$ for each homology generator of $H_*((X_2)^{n/2})$. This means

$$(\gamma|_{S_a})_*(q) = (\sigma_a \circ \pm G^{n/2} \circ \sigma_a^{-1})_*(q) = \pm(\gamma_a)_*(q)$$

for each homology generator $q \in H_*(S_a)$. As a result the complex vector subspace $\gamma_*(H_*(X_n))$ in $H_*((S^2)^n)$ equals the subspace of $H_*((S^2)^n)$ spanned by $\bigcup_a ((\gamma_a)_*(H_*(S_a)))$. \square

3.2. The S_n action on $H_*(X_n)$. There is a natural S_n action on $(S^2)^n$ that permutes coordinates, in which $\sigma \in S_n$ acts on $(x_1, \dots, x_n) \in (S^2)^n$ by $\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$.

This section proves that the induced S_n action on $H_*((S^2)^n)$ restricts to the image $\gamma_*(H_*(X_n))$. The proof computes the action of each simple transposition s_i on each homology generator of $H_*(X_n)$ and shows it is a linear combination of elements of $H_*(X_n)$.

We note that this is not true for most embeddings $X_n \rightarrow (S^2)^n$. In fact similar computations to Lemma 3.2 show it fails for the identity embedding even when $n = 4$.

Both γ_* and the collection $\{(\gamma_a)_*\}$ map to the same vector subspace of $H_*((S^2)^n)$, but they may differ by a sign on homology generators. To establish a sign convention, we choose a set of generators of $\gamma_*(H_*(X_n))$ as follows.

Definition 3.8. *Given a standard dotted noncrossing matching $M \in H_*(X_n)$ and the component S_a for which $M \in H_*(S_a)$, define the generator of $\gamma_*(H_*(X_n))$ corresponding to M to be $(\gamma_a)_*(M)$.*

The generator $(\gamma_a)_*(M)$ has many useful combinatorial properties which the following definitions will help make explicit.

Definition 3.9. *The undot set of a line diagram is the set of all vertices which have undotted line segments. For example, the line diagrams in Figure 3.1 have undot sets $\{2, 4\}$ and $\{1, 2, 3\}$.*

The undot sets provide a bijective correspondence between line diagrams on n nodes and subsets of $\{1, \dots, n\}$. If U is such a subset, denote by l_U the line diagram with undot set U .

Each standard dotted noncrossing matching determines a *line diagram sum*.

Definition 3.10. *Let M be a standard dotted noncrossing matching on n vertices. Define \mathcal{U}_M to be the collection of undot sets given by*

$$\mathcal{U}_M = \{U \subseteq \{1, 2, \dots, n\} : U \text{ contains exactly one endpoint of each undotted arc in } M\}.$$

For each set $U \in \mathcal{U}_M$, let $\Lambda_M(U)$ be the number of left endpoints in U . Define the line diagram sum L_M of M to be

$$L_M = \sum_{U \in \mathcal{U}_M} (-1)^{\Lambda_M(U)} l_U.$$

We give an example of \mathcal{U}_M and L_M for $M = \frown \smile$. The matching M has undotted arc

$(1, 2)$ so $\mathcal{U}_M = \{\{1\}, \{2\}\}$ and L_M is $\bullet \mid \bullet \mid - \bullet \mid \bullet \mid$.

Lemma 3.11. *Suppose the standard dotted noncrossing matching M' is obtained from M by inserting an arc (i, j) with $1 \leq i < j \leq n + 2$. If (i, j) is a dotted arc then*

$$L_{M'} = \sum_{U \in \mathcal{U}_M} (-1)^{\Lambda_M(U)} l_{\widetilde{U}_\emptyset}$$

and if (i, j) is an undotted arc then

$$L_{M'} = \sum_{U \in \mathcal{U}_M} (-1)^{\Lambda_M(U)} l_{\widetilde{U}_j} - \sum_{U \in \mathcal{U}_M} (-1)^{\Lambda_M(U)} l_{\widetilde{U}_i}$$

where if $S \subseteq \{i, j\}$ then \widetilde{U}_S denotes the undot set obtained from U by

$$\widetilde{U}_S = S \cup \{x \in U : x < i\} \cup \{x + 1 : x \in U \text{ and } i \leq x < j\} \cup \{x + 2 : x \in U \text{ and } j \leq x\}.$$

Proof. Assume M' is obtained from M by inserting the arc (i, j) for $1 \leq i < j \leq n + 2$. Define the function $c : \{1, \dots, n\} \rightarrow \{1, \dots, n + 2\}$ by

$$c(x) = \begin{cases} x & \text{if } x < i, \\ x + 1 & \text{if } i \leq x < j, \text{ and} \\ x + 2 & \text{if } j \leq x. \end{cases}$$

First assume (i, j) is a dotted arc. No additional undotted arcs have been added to the matching M' so $|\mathcal{U}_M| = |\mathcal{U}_{M'}|$. Let $U \in \mathcal{U}_M$ and let $\widetilde{U}_\emptyset = \{c(x) : x \in U\}$. The line diagram $l_{\widetilde{U}_\emptyset}$ is obtained from l_U by inserting one dotted segment between vertices $i - 1$ and i and

another between vertices $j - 1$ and j . For each $U \in \mathcal{U}_M$ we have $\Lambda_M(U) = \Lambda_{M'}(\widetilde{U}_\emptyset)$. We find

$$L_{M'} = \sum_{U \in \mathcal{U}_M} (-1)^{\Lambda_M(U)} l_{\widetilde{U}_\emptyset}.$$

Now assume (i, j) is undotted. One undotted arc has been added to M' so $|\mathcal{U}_{M'}| = 2 \cdot |\mathcal{U}_M|$.

For each $U \in \mathcal{U}_M$ we construct two distinct undot sets $\widetilde{U}_i, \widetilde{U}_j \in \mathcal{U}_{M'}$. Let $\widetilde{U}_i = \widetilde{U}_\emptyset \cup \{i\}$ and $\widetilde{U}_j = \widetilde{U}_\emptyset \cup \{j\}$. The line diagram $l_{\widetilde{U}_i}$ is obtained from l_U by inserting an undotted segment between vertices $i - 1$ and i and a dotted segment between vertices $j - 1$ and j ; the diagram $l_{\widetilde{U}_j}$ reverses the roles of i and j . Hence $\Lambda_M(U) = \Lambda_{M'}(\widetilde{U}_j)$ and $\Lambda_M(U) + 1 = \Lambda_{M'}(\widetilde{U}_i)$ and

$$L_{M'} = \sum_{U \in \mathcal{U}_M} (-1)^{\Lambda_M(U)} l_{\widetilde{U}_j} - \sum_{U \in \mathcal{U}_M} (-1)^{\Lambda_M(U)} l_{\widetilde{U}_i}.$$

□

Lemma 3.12. *If $M \in H_*(S_a)$ is a standard dotted noncrossing matching $(\gamma_a)_*(M) = L_M$.*

Proof. Lemma 3.2 proves the case when $n = 2$.

Assume as the inductive hypothesis that if M is any standard dotted noncrossing matching on n vertices then $(\gamma_a)_*(M) = L_M$. Let a' be a matching obtained from a by inserting an arc (i, j) with $1 \leq i < j \leq n + 2$. By construction the map $(\gamma_{a'})_*$ is a permutation of $(\gamma_a)_* \otimes g_*$ so that g_* is applied to coordinates i and j and $(\gamma_a)_*$ is applied to the other coordinates (in their original order).

Let M be a dotting of a and construct a dotted noncrossing matching M' by adding the dotted arc (i, j) . From the $n = 2$ case we know $g_*([p]) = [p \times p]$. The homology generator $[p] \in H_*(S^2)$ is denoted by a dotted vertical line segment so the image $(\gamma_{a'})_*(M')$ agrees with $(\gamma_a)_*(M)$ except that dotted segments are inserted into the i^{th} and j^{th} positions. This is exactly the change to the line diagram sum described in Lemma 3.11 as $l_{\widetilde{U}_\emptyset}$.

Now let M' be obtained from M by inserting an undotted arc (i, j) . We know $g_*([c]) = [p \times c] - [c \times p]$. Hence each term in the image of M appears twice in the image of M' : one occurrence of M has an undotted i^{th} segment and dotted j^{th} segment with sign changed; the other has a dotted i^{th} segment and undotted j^{th} segment with no change in sign. This is exactly the formula described in Lemma 3.11. □

Corollary 3.13. *The map $\gamma_* : H_*(X_n) \rightarrow H_*((S^2)^n)$ is injective.*

Proof. Let U_M be the undot set obtained from M by choosing the right endpoint of each undotted arc. Create a Young tableau from U_M by putting the numbers from U_M on the bottom row and all other numbers on the top row (with both rows in increasing order). Theorem 2.15 showed that the standard noncrossing matchings M with exactly k undotted arcs are bijective with the tableaux associated to M , and hence so are the sets U_M .

Order subsets of $\{1, 2, \dots, n\}$ of cardinality k according to the rule that $S = \{i_1 < i_2 < \dots < i_k\}$ is less than $S' = \{i'_1 < i'_2 < \dots < i'_k\}$ if for some r both $i_r < i'_r$ and $i_{r+1} = i'_{r+1}$, $i_{r+2} = i'_{r+2}$, \dots , $i_k = i'_k$. Note that U_M is the maximum set in \mathcal{U}_M with respect to this order.

If the basis vectors $l_U \in H_*((S^2))^n$ are ordered according to their undot sets then the map $M \mapsto L_M$ is represented by a matrix in row-echelon form; the leading term in each row is the coefficient of l_{U_M} , namely $\Lambda_M(U_M) = (-1)^0 = 1$. Hence $\{L_M\}$ is linearly independent in $H_*((S^2)^n)$. The standard dotted noncrossing matchings M generate $H_*(X_n)$ so γ_* is injective. \square

Because homology generators for $H_*((S^2)^n)$ consist of a choice of point or two-cell for each coordinate, the S_n action permutes the choices of point or two-cell in the same way it permutes coordinates. On the level of the line diagrams, this action permutes line segments. Figure 3.4 has an example.

$$(123) \cdot \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & \bullet & \bullet \\ \hline \end{array} = \begin{array}{|c|c|c|c|} \hline \bullet & \bullet & \bullet & \bullet \\ \hline \end{array}$$

FIGURE 3.4. Permutation action in $H_*((S^2)^4)$.

Let s_i be the simple transposition exchanging i and $i + 1$. Choose a homology generator $M \in H_*(S_a)$. Since γ_* is injective and $\gamma_*(M) = \pm(\gamma_a)_*(M)$, we may identify the image $(\gamma_a)_*(M)$ with M . We will show that $s_i(M) \in H_*(X_n)$, namely the S_n action on $H_*((S^2)^n)$ restricts to $\gamma_*(H_*(X_n))$.

Theorem 3.14. *The chart in Figure 3.6 describes the action of simple transpositions on $\gamma_*(H_*(X_n))$. The S_n - action on $H_*(X_n)$ defined so that $\sigma \in S_n$ acts on a generator M by*

$$\sigma \cdot M := \sigma \cdot L_M$$

is a well-defined representation.

Proof. The action of S_n on $(S^2)^n$ is consistent with the S_n -actions on sets and line diagrams given as follows. If U is an undot set then $\sigma \in S_n$ acts on U by $\sigma \cdot U = \{\sigma \cdot x : x \in U\}$ and $\sigma \cdot \mathcal{U}_M = \{\sigma \cdot U \text{ for } U \in \mathcal{U}_M\}$. By definition $\sigma \cdot l_U = l_{\sigma \cdot U}$ and

$$\sigma \cdot L_M = \sum_{U \in \mathcal{U}_M} (-1)^{\Lambda_M(U)} l_{\sigma \cdot U}.$$

The simple transpositions s_i generate the symmetric group; to prove that S_n restricts to an action on $\gamma_*(H_*(X_n))$ it suffices to show that for each line diagram sum L_M the image $s_i(L_M)$ is a linear combination of line diagram sums. We do this by explicitly calculating $s_i(L_M)$ in the four possible configurations of arcs and dottings: if i and $i + 1$ are both incident to dotted arcs in M (whether or not $(i, i + 1)$ is an arc); if $(i, i + 1)$ is an undotted arc in M ; if i and $i + 1$ are incident to different arcs in M , exactly one of which is dotted; and if i and $i + 1$ are incident to different arcs in M , neither of which is dotted. Since the map

$M \mapsto L_M$ is injective, the preimage of each $s_i(L_M)$ is unique and the S_n -action on $H_*(X_n)$ is a well-defined S_n -representation.

Figure 3.6 gives each case diagrammatically. As in Lemma 3.5, the chart only shows arcs incident to vertices i and $i + 1$, since any other arcs are common to M and M' .

Case 1: Every arc in the matching M incident to i and $i + 1$ is dotted.

This case encompasses both the case when $(i, i + 1)$ is a dotted arc in M and the case when (j, i) and $(i + 1, k)$ are both dotted arcs in M . For all such M no $U \in \mathcal{U}_M$ contains either i or $i + 1$. Hence $s_i \cdot U = U$ for all $U \in \mathcal{U}_M$. By definition $s_i \cdot L_M = L_M$ in this case.

Case 2: The matching M contains the undotted arc $(i, i + 1)$.

If $(i, i + 1)$ is an undotted arc then let \mathcal{U}_i be the collection of all undot sets in \mathcal{U}_M containing i and let \mathcal{U}_{i+1} be the undot sets containing $i + 1$. By definition \mathcal{U}_i and \mathcal{U}_{i+1} are disjoint and $\mathcal{U}_M = \mathcal{U}_i \cup \mathcal{U}_{i+1}$. The map $s_i : \mathcal{U}_i \rightarrow \mathcal{U}_{i+1}$ is a bijection with inverse s_i . In particular $s_i \cdot \mathcal{U}_M = \mathcal{U}_M$. The vertex i is a left endpoint and $i + 1$ is a right endpoint so

$$(-1)^{\Lambda_M(U)-1} = (-1)^{\Lambda_M(s_i \cdot U)}$$

if $U \in \mathcal{U}_M$. It follows that

$$\begin{aligned} s_i \cdot L_M &= \sum_{U \in \mathcal{U}_i} (-1)^{\Lambda_M(U)} l_{s_i \cdot U} + \sum_{U \in \mathcal{U}_{i+1}} (-1)^{\Lambda_M(U)} l_{s_i \cdot U} \\ &= \sum_{U \in \mathcal{U}_i} (-1)^{\Lambda_M(U)-1} l_U + \sum_{U \in \mathcal{U}_{i+1}} (-1)^{\Lambda_M(U)-1} l_U = -L_M. \end{aligned}$$

Note: Both Case 3 and Case 4 assume the standard matching M contains arcs (j, i) and $(i + 1, k)$ with no additional hypotheses on the positions of j and k . All three possible relative positions of the arcs (j, i) and $(i + 1, k)$, shown in Figure 3.5, are consistent with our calculations.

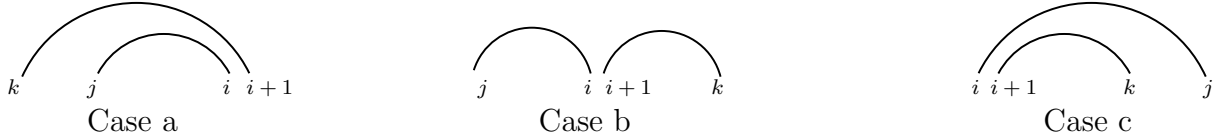


FIGURE 3.5. Arcs incident on vertices i and $i + 1$. (We allow additional arcs whose endpoints are consistent with those in the diagram.)

Case 3: The standard matching M contains the arcs (j, i) and $(i + 1, k)$, exactly one of which is dotted. (There are no additional hypotheses on j and k .)

Assume arc (j, i) is undotted and arc $(i + 1, k)$ is dotted; the case when (j, i) is dotted is symmetric. Let \mathcal{U}_j be the undot sets in \mathcal{U}_M containing j and let \mathcal{U}_i be the undot sets containing i . As in Case 1 there is a disjoint union $\mathcal{U}_M = \mathcal{U}_j \cup \mathcal{U}_i$. Let M' be the standard matching with dotted arc (j, k) and undotted arc $(i, i + 1)$ that is otherwise identical to M . Then $\mathcal{U}_{M'} = \mathcal{U}'_i \cup \mathcal{U}'_{i+1}$.

Matchings M and M' each have an undotted arc incident to vertex i ; the other undotted arcs in M are exactly the same as those in M' . Thus $\mathcal{U}_i = \mathcal{U}'_i$. The matching M is standard

so $j < i$. For each $U \in \mathcal{U}_i$ we have $\Lambda_M(U) = \Lambda_{M'}(U) - 1$ since i is a right endpoint in M and a left endpoint in M' . The maps $s_i : \mathcal{U}_j \rightarrow \mathcal{U}_j$ and $s_i : \mathcal{U}_i \rightarrow \mathcal{U}'_{i+1}$ are bijections. For each $U \in \mathcal{U}_i$ we have $\Lambda_M(U) = \Lambda_{M'}(s_i \cdot U)$ since i is a right endpoint in M and $i + 1$ is a right endpoint in M' . Together this gives

$$\begin{aligned}
 s_i \cdot L_M &= \sum_{U \in \mathcal{U}_j} (-1)^{\Lambda_M(U)} l_{s_i \cdot U} + \sum_{U \in \mathcal{U}_i} (-1)^{\Lambda_M(U)} l_{s_i \cdot U} \\
 &= \sum_{U \in \mathcal{U}_j} (-1)^{\Lambda_M(U)} l_U + \sum_{U \in \mathcal{U}'_{i+1}} (-1)^{\Lambda_{M'}(U)} l_U \\
 &= \sum_{U \in \mathcal{U}_j} (-1)^{\Lambda_M(U)} l_U + \sum_{U \in \mathcal{U}'_{i+1}} (-1)^{\Lambda_{M'}(U)} l_U + \sum_{U \in \mathcal{U}_i} (-1)^{\Lambda_M(U)} l_U - \sum_{U \in \mathcal{U}_i} (-1)^{\Lambda_{M'}(U)} l_U \\
 &= \sum_{U \in \mathcal{U}_j} (-1)^{\Lambda_M(U)} l_U + \sum_{U \in \mathcal{U}_i} (-1)^{\Lambda_M(U)} l_U + \sum_{U \in \mathcal{U}'_i} (-1)^{\Lambda_{M'}(U)} l_U + \sum_{U \in \mathcal{U}'_{i+1}} (-1)^{\Lambda_{M'}(U)} l_U \\
 &= L_M + L_{M'}.
 \end{aligned}$$

Case 4: The standard matching M contains the undotted arcs (j, i) and $(i + 1, k)$. (There are no additional hypotheses on j and k .)

Suppose both (j, i) and $(i + 1, k)$ are undotted. For each subset $S \subseteq \{j, i, i + 1, k\}$ let \mathcal{U}_S denote the collection of $U \in \mathcal{U}$ with $S \subseteq U$. There is a disjoint union

$$\mathcal{U}_M = \mathcal{U}_{j,i+1} \cup \mathcal{U}_{j,k} \cup \mathcal{U}_{i,i+1} \cup \mathcal{U}_{i,k}.$$

Let M' be the matching with undotted arcs (j, k) and $(i, i + 1)$ that is otherwise identical to M . Then $\mathcal{U}_{M'} = \mathcal{U}'_{j,i} \cup \mathcal{U}'_{j,i+1} \cup \mathcal{U}'_{i,k} \cup \mathcal{U}'_{k,i+1}$. The sets $\mathcal{U}_{j,i+1} = \mathcal{U}'_{j,i+1}$ and $\mathcal{U}_{i,k} = \mathcal{U}'_{i,k}$ since M and M' agree off of vertices j, k, i , and $i + 1$. The maps $s_i : \mathcal{U}_{i,k} \rightarrow \mathcal{U}'_{k,i+1}$ and $s_i : \mathcal{U}_{j,i} \rightarrow \mathcal{U}'_{j,i+1}$ are bijections while $s_i : \mathcal{U}_{j,k} \rightarrow \mathcal{U}_{j,k}$ and $s_i : \mathcal{U}_{i,i+1} \rightarrow \mathcal{U}_{i,i+1}$ are identity maps.

In all cases shown in Figure 3.5, for each $U \in \mathcal{U}_{j,i+1}$ we have $\Lambda_M(U) = \Lambda_{M'}(s_i \cdot U)$ and for each $U \in \mathcal{U}_{i,k}$ we have $\Lambda_M(U) = \Lambda_{M'}(s_i \cdot U)$. If $U \in \mathcal{U}_{i,k}$ or $U \in \mathcal{U}_{j,i+1}$ then s_i changes exactly one endpoint in U from right to left or vice versa, namely $\Lambda_M(U) = \Lambda_{M'}(U) \pm 1$. This gives

$$\begin{aligned}
 s_i \cdot L_M &= \sum_{U \in \mathcal{U}_{j,i+1}} (-1)^{\Lambda_M(U)} l_{s_i \cdot U} + \sum_{U \in \mathcal{U}_{j,k}} (-1)^{\Lambda_M(U)} l_{s_i \cdot U} + \sum_{U \in \mathcal{U}_{i,i+1}} (-1)^{\Lambda_M(U)} l_{s_i \cdot U} + \sum_{U \in \mathcal{U}_{i,k}} (-1)^{\Lambda_M(U)} l_{s_i \cdot U} \\
 &= \sum_{U \in \mathcal{U}'_{j,i}} (-1)^{\Lambda_{M'}(U)} l_U + \sum_{U \in \mathcal{U}_{j,k}} (-1)^{\Lambda_M(U)} l_U + \sum_{U \in \mathcal{U}_{i,i+1}} (-1)^{\Lambda_M(U)} l_U + \sum_{U \in \mathcal{U}'_{i+1,k}} (-1)^{\Lambda_{M'}(U)} l_U \\
 &\quad + \sum_{U \in \mathcal{U}_{j,i+1}} (-1)^{\Lambda_M(U)} l_U - \sum_{U \in \mathcal{U}_{j,i+1}} (-1)^{\Lambda_M(U)} l_U + \sum_{U \in \mathcal{U}_{i,k}} (-1)^{\Lambda_M(U)} l_U - \sum_{U \in \mathcal{U}_{i,k}} (-1)^{\Lambda_M(U)} l_U \\
 &= L_M + L_{M'}.
 \end{aligned}$$

□



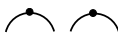
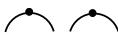


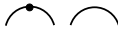
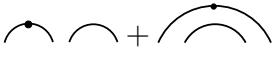

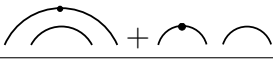
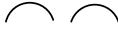
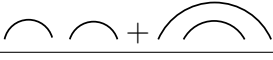

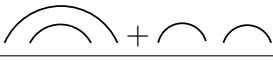
Vertex Labelings	M	$s_i \cdot M$
Case 1 $i, i+1$ both on dotted arcs		
		
Case 2 $(i, i+1)$ is an undotted arc		$-$ 
Case 3 (j, i) and $(i+1, k)$ have one dot (the dotted arc is on right or left depending on the sign of $i-j$)		
		
Case 4 (j, i) and $(i+1, k)$ have no dots		
		

FIGURE 3.6. The S_n -action on standard dotted noncrossing matchings.

4. THE S_n REPRESENTATION ON $H_k(X_n)$

This section explicitly identifies the S_n representation on the homology of the Springer fiber in each degree. First we review Specht modules, a classical construction of irreducible representations of S_n . Then we give a direct proof that the \mathbb{C} -module $\gamma_*(H_k(X_n))$ with the S_n -action from the antipodal embedding is the Specht module of the partition $(n-k, k)$.

4.1. Specht modules for partitions $(n-k, k)$. We sketch the classical construction of Specht modules both in general and for two-row partitions; [5, Chapter 7] has more.

Let λ be a partition of n considered as a Young diagram (a collection of left- and top-aligned boxes so that the i^{th} row has λ_i boxes for each i). A Young tableau is a filling of the Young diagram λ with the numbers $1, 2, \dots, n$ without repetition. A *tabloid* T is an equivalence class of Young tableaux, where the equivalence relation is reordering of the numbers in each row. The permutation group S_n acts on the collection of tabloids; the action of $w \in S_n$ on the tabloid T replaces i with $w(i)$ for each $i = 1, \dots, n$. For each *tableau* T let $\text{Col}(T)$ denote the subgroup of permutations in S_n that stabilize each column of T .

Consider the complex vector space with one basis element v_T for each tabloid T of shape λ . The Specht module V_λ is the subspace spanned by the vectors

$$e_T = \sum_{w \in \text{Col}(T)} \text{sign}(w) v_{w(T)}$$

for each *tableau* T of shape λ . Figure 4.1 gives an example. A classical theorem states that V_λ is the unique (up to isomorphism) irreducible representation of S_n corresponding to λ . A small exercise shows that V_λ has a basis $\{e_T\}$ indexed by *standard* Young tableaux, namely tableaux whose rows increase left-to-right and whose columns increase top-to-bottom.

The construction of the Specht module is simpler when the partition has two rows, as in Figure 4.1. The subgroup $\text{Col}(T)$ for a two-row standard Young tableau T is a product of transpositions, one for each column of height two. The sign of the permutation $w \in \text{Col}(T)$ is determined by the parity of the number of columns in which w exchanges entries.

Tableau T	Basis element e_T
$\begin{array}{ c c } \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} - \begin{array}{ c c } \hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} - \begin{array}{ c c } \hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} + \begin{array}{ c c } \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$
$\begin{array}{ c c } \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$	$\begin{array}{ c c } \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} - \begin{array}{ c c } \hline 2 & 3 \\ \hline 1 & 4 \\ \hline \end{array} - \begin{array}{ c c } \hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} + \begin{array}{ c c } \hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}$

FIGURE 4.1. Generators of the Specht module for the partition $(2, 2)$

4.2. The S_n action on the homology of X_n . The standard noncrossing matchings naturally define another subspace W_λ of the complex vector space spanned by

$$\{v_T : T \text{ is a tabloid of shape } (n - k, k)\}.$$

We define a set of generators for W_λ and then show that in fact $V_\lambda = W_\lambda$.

Definition 4.1. *Let M be a standard noncrossing matching with exactly k undotted arcs and let $T = \varphi(M)$. Define the subgroup $\text{Undot}(M) \subseteq S_n$ to be the collection of permutations generated by the transpositions (ij) for each undotted arc (i, j) . The matching module W_λ is the subspace spanned by the vectors*

$$e_M = \sum_{w \in \text{Undot}(M)} \text{sign}(w) v_{w(T)}.$$

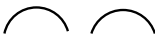

Matching M	Tableau $\varphi(M)$	Basis element e_M																				
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FIGURE 4.2. Generators of the matching module for the partition $(2, 2)$

Figure 4.2 gives these generators for the standard noncrossing undotted matchings when $n = 4$. Note that the second generator in Figure 4.2 is not the same as in Figure 4.1.

Lemma 4.2. *The map $M \mapsto e_M$ induces an S_n -equivariant isomorphism of complex vector spaces $H_{2k}(X_n) \cong W_{(n-k, k)}$.*

Proof. We first define an S_n -equivariant \mathbb{C} -linear isomorphism ψ that sends line diagrams to tabloids. We then show $M \mapsto e_M$ factors S_n -equivariantly through ψ .

Let l be the line diagram for a basis element of $H_*((S^2)^n)$. Assign the line diagram l to the tabloid $\psi(l)$ which has i on the top row if the i^{th} line is dotted and on the second row otherwise. Line diagrams are in bijection with their undot sets, and k -element undot sets are in bijection with second rows of tabloids for the partition $(n - k, k)$. Thus ψ is a bijection between the basis for $H_{2k}((S^2)^n)$ and the basis for the space of tabloids for the partition $(n - k, k)$.

Extending by \mathbb{C} -linearity gives a map from $H_{2k}((S^2)^n)$ to the vector space of tabloids for the partition $(n - k, k)$. The permutation w sends the i^{th} strand of l to the $w(i)^{\text{th}}$ position. Hence $\psi(w(l))$ has $w(i)$ on the top row if and only if the i is on the top row in $\psi(l)$ so the map ψ is S_n -equivariant.

We show that if L_M is the line diagram for M then $\psi(L_M) = e_M$. If M has a dotted arc (i, j) then both i and j are dotted lines in every term of L_M , and both i and j are on the top row of every tabloid $w(\varphi(M))$ for $w \in \text{Undot}(M)$.

If M has no undotted arcs, the line diagram sum L_M has a single term with all n lines dotted. The group $\text{Undot}(M)$ is trivial so e_M is the standard tableaux with a single row of n boxes. Therefore $\psi(L_M) = e_M$ and the base case is proven.

Assume as the inductive hypothesis that $\psi(L_M) = e_M$ if M has at most $k - 1$ undotted arcs. Suppose M' is obtained by inserting an undotted arc (i, j) into M , renumbering vertices as needed. The line diagram $L_{M'}$ has two terms for each term l of L_M . In the first term, the i^{th} strand is undotted, the j^{th} strand is dotted, and the sign differs with that of l . In the second term, the i^{th} strand is dotted, the j^{th} strand is undotted, and the sign agrees with that of l . The tableau $\varphi(M')$ agrees with $\varphi(M)$ except that i is inserted into the top row, j into the bottom row, and all other entries are renumbered accordingly.

Given $w \in \text{Undot}(M)$, let $w' \in \text{Undot}(M')$ be the permutation obtained by renumbering to allow for the insertion of the arc (i, j) . Then $e_{M'}$ has two terms for each term $v_{w(\varphi(M))}$ of e_M , namely $\text{sign}(w')v_{w'(\varphi(M'))}$ and $\text{sign}(s_{ij}w')v_{s_{ij}w'(\varphi(M'))}$. These are exactly the images under ψ of the new terms in the line diagram. By induction on the number of undotted arcs we conclude $\psi(L_M) = e_M$ for each standard noncrossing matching M .

We conclude that ψ restricts to an isomorphism of $\gamma_*(H_{2k}(X_n))$ with $W_{(n-k,k)}$. The map $M \mapsto L_M$ is injective so the map $M \mapsto e_M$ is an isomorphism of $H_{2k}(X_n)$ with $W_{(n-k,k)}$. The S_n -action on matchings is defined so that the map $M \mapsto L_M$ is S_n -equivariant; since ψ is S_n -equivariant, the map $M \mapsto e_M$ is an S_n -equivariant isomorphism. \square

Theorem 4.3. *Let V_λ denote the Specht module corresponding to λ . Let W_λ denote the module defined by the standard noncrossing matchings. Then V_λ equals W_λ as S_n -representations. In particular the S_n action on $H_*(X_n)$ is the Springer representation.*

Proof. Both V_λ and W_λ are S_n -representations in the vector space of tabloids. This means that their intersection (in the vector space generated by tabloids) is also an S_n representation. We know that V_λ is an irreducible representation, so the only subspaces of V_λ that are preserved by the S_n action are 0 and V_λ . Hence $V_\lambda \cap W_\lambda$ is either 0 or V .

Let M_0 be the unnnested standard noncrossing matching with $n-k$ undotted arcs on the left and k dotted arcs on the right. Let T_0 be the standard tableau with $2, 4, 6, 8, 10, \dots, 2n-2k$ in the second row. The tableau T_0 corresponds to M_0 under the bijection of Section 2. In this case the columns are exactly the undotted arcs and so $e_{M_0} = e_{T_0}$.

This shows that $V_\lambda \cap W_\lambda$ contains at least one nonzero vector, and hence is V_λ . Since V_λ and W_λ have the same dimension, we conclude that $V_\lambda = V_\lambda \cap W_\lambda = W_\lambda$. By Proposition 2.5 this is the Springer representation. \square

Conjecture 4.4. *The Kazhdan-Lusztig basis for irreducible representations of S_n is defined to be the geometric basis induced by the representation on the top-degree cohomology of the Springer fiber. This is the basis we identified in top degree. We conjecture that in fact the matching basis is the Kazhdan-Lusztig basis in all degrees.*

5. APPENDIX

We review the relevant results from [2] and prove that Khovanov's construction of the Springer variety is diffeomorphic to the classical construction of the Springer variety.

Our notation uses that of [2] except where it conflicts with notation already used in this paper. Let X be a linear operator $X : \mathbb{C}^{2N} \rightarrow \mathbb{C}^{2N}$ with Jordan blocks of size (N, N) . Fix a basis $e_1, \dots, e_N, f_1, \dots, f_N$ for \mathbb{C}^{2N} so that $X(e_i) = e_{i-1}$ and $X(f_i) = f_{i-1}$ for each $i \geq 2$, and so that $\ker X$ is the span of e_1, f_1 . We use the Hermitian inner product with orthonormal basis $e_1, e_2, \dots, e_N, f_1, f_2, \dots, f_N$. Assume that $N \geq 2n$ and define a partial Springer variety of nested vector subspaces of \mathbb{C}^{2N} by

$$Y_n = \{(V_1 \subseteq V_2 \subseteq \dots \subseteq V_n) : \text{each } V_i \text{ is } i\text{-dimensional and } XV_i \subseteq V_{i-1}\}.$$

Recall that $B^{n/2}$ is the set of all noncrossing matchings on n vertices. Define

$$X'_n = \bigcup_{a \in B^{n/2}} S'_{a,n}$$

where

$$S'_{a,n} = \{(x_1, x_2, \dots, x_n) \in (S^2)^n : x_i = -x_j \text{ if } (i, j) \in a\}.$$

(As before $-x$ denotes the image of x under the antipodal map.)

[2, Theorem 2.1] proves

Proposition 5.1. (Cautis-Kamnitzer) *Choose an orthonormal basis e, f for \mathbb{P}^1 and define a map $C : \mathbb{P}^{2N-1} \rightarrow \mathbb{P}^1$ so that*

$$\text{span}_{\mathbb{C}} \left(\sum a_i e_i + \sum b_i f_i \right) \mapsto \text{span}_{\mathbb{C}} \left(\left(\sum a_i \right) e + \left(\sum b_i \right) f \right).$$

For each $V_{\bullet} \in Y_n$ define lines L_1, \dots, L_n by $V_i = V_{i-1} \oplus L_i$ and $L_i \perp V_{i-1}$. Then the map $\ell : Y_n \rightarrow (\mathbb{P}^1)^n$ defined by

$$V_{\bullet} \mapsto (C(L_1), C(L_2), \dots, C(L_n))$$

is a diffeomorphism.

Furthermore define $Z_n^i = \{V_{\bullet} \in Y_n : XV_{i+1} = V_{i-1}\}$. Then the image of Z_n^i under the diffeomorphism ℓ is exactly the elements in $(\mathbb{P}^1)^n$ satisfying $C(L_i) = -C(L_{i+1})$.

For each noncrossing matching a and each n let $C_{a,n}$ denote the preimage $\ell^{-1}(S'_{a,n})$.

Let a be a noncrossing matching on n vertices containing the arc $(i, i+1)$ and let a' be a noncrossing matching on $n-2$ vertices. Suppose a' is obtained from a by erasing the arc $(i, i+1)$, so that if $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n-2\}$ is the map

$$f(j) = \begin{cases} j & \text{if } j \leq i-1 \text{ and} \\ j-2 & \text{if } j \geq i+2 \end{cases}$$

then $a' = \{(f(j), f(j')) : (j, j') \in a' \text{ and } (j, j') \neq (i, i+1)\}$. Define the projection map $q' : S'_{a,n} \rightarrow S'_{a',n-2}$ by $q'(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, \widehat{x_i}, \widehat{x_{i+1}}, \dots, x_n)$ where $\widehat{x_j}$ omits the j^{th}

coordinate.

In [2, Section 2.2] Cautis-Kamnitzer define a map $q : Z_n^i \rightarrow Y_{n-2}$ by

$$V_{\bullet} \mapsto (V_1, \dots, V_{i-1}, XV_{i+2}, XV_{i+3}, \dots, XV_n)$$

and prove that Z_n^i is a \mathbb{P}^1 -bundle over Y_{n-2} .

Lemma 5.2. *Let a be a noncrossing matching on n vertices containing the arc $(i, i+1)$ and let a' be a noncrossing matching on $n-2$ vertices. Suppose a is obtained from a' by erasing the arc $(i, i+1)$. Then there is a commutative diagram*

$$\begin{array}{ccc} C_{a,n} & \xrightarrow{\ell} & S'_{a,n} \\ q \downarrow & & \downarrow q' \\ Y_{n-2} & \xrightarrow{\ell} & S'_{a',n-2} \end{array}$$

and the image $q(C_{a,n})$ is $C_{a',n-2}$.

Proof. We will show $q' \circ \ell = \ell \circ q$. Then since ℓ is a diffeomorphism and $q' \circ \ell$ is surjective, the image $q(C_{a,n})$ is diffeomorphic to $S'_{a',n}$. By definition $q(C_{a,n}) = C_{a',n-2}$.

By construction the i^{th} and $(i+1)^{\text{th}}$ coordinates of $S'_{a,n}$ are antipodes of each other so

$C_{a,n} \subseteq Z_n^i$. If $V_\bullet \in Z_n^i$ then $V_{i+1} = X^{-1}V_{i-1}$ and in particular $\ker X \subseteq V_{i+1}$. Since the line L_j is defined so that $L_j \perp V_{j-1}$ and $V_j = L_j \oplus V_{j-1}$ for each j , we conclude that $L_j \perp \ker X$ for $j \geq i+2$. Thus L_j is spanned by $\sum_{i \geq 2} a_i e_i + b_i f_i$ for some coefficients a_i, b_i .

So $C(L_j) = C(XL_j)$ for all $j \geq i+2$.

We confirm that $XL_j \perp XV_{j-1}$ for $j \geq i+2$. The line L_j is the span of $\sum_{i \geq 2} a_i e_i + b_i f_i$

so if $w \in V_{j-1}$ then since e_i, f_i are an orthonormal basis the inner product satisfies

$$\left\langle \sum_{i \geq 2} a_i e_i + b_i f_i, w \right\rangle = \left\langle \sum_{i \geq 2} a_i e_{i-1} + b_i f_{i-1}, Xw \right\rangle.$$

This shows that if $V_\bullet \in C_{a,n}$ then

$$\begin{aligned} \ell(q(V_\bullet)) &= (C(L_1), C(L_2), \dots, C(L_{i-1}), C(XL_{i+2}), C(XL_{i+3}), \dots, C(XL_n)) \\ &= (C(L_1), C(L_2), \dots, C(L_{i-1}), C(L_{i+2}), C(L_{i+3}), \dots, C(L_n)) \\ &= q'(\ell(V_\bullet)). \end{aligned}$$

□

The next lemma proves that the $C_{a,n}$ are the components of the Springer variety X_n .

Lemma 5.3. *The union $\bigcup_{a \in B^{n/2}} C_{a,n}$ is isomorphic to the $(n/2, n/2)$ Springer variety in the vector space $\ker X^{n/2}$. Each $C_{a,n}$ is an irreducible component of the Springer variety.*

Proof. The proof inducts on n . When $n = 2$ there is a unique crossingless matching $a' = (1, 2)$. In this case $C_{a',2} = Z_2^1 = \{V_\bullet \in Y_2 : V_2 = X^{-1}V_0\}$. Equivalently

$$C_{a',2} = \{V_1 \subseteq V_2 : V_2 = \ker X \text{ and } XV_j \subseteq V_j \text{ for all } j\},$$

which means $C_{a',2}$ is isomorphic to the $(1, 1)$ Springer variety.

Assume that the claim holds for $n - 2$. Let (i, j) be an arc in a with minimal distance $j - i$; if $j - i \neq 1$ then a is not a noncrossing matching. Let a' be the matching with $(i, i + 1)$

removed, so $C_{a,n} \xrightarrow{q} C_{a',n-2}$. If $q(V_\bullet) = V'_\bullet$ then for all j there exists k with $V_j \subseteq X^{-1}V'_k$.

By the inductive hypothesis each $V'_k \subseteq \ker X^{(n-2)/2}$ and so each $V_j \subseteq \ker X^{n/2}$. Thus for all noncrossing matchings a , each $C_{a,n}$ is contained in the $(n/2, n/2)$ Springer variety inside

$\ker X^{n/2}$. Since ℓ is a diffeomorphism we conclude that all $C_{a,n}$ are compact irreducible subvarieties of the Springer variety of dimension $\dim X_n$, namely each $C_{a,n}$ is a component of the Springer variety. Moreover if $a \neq b$ then $C_{a,n} \neq C_{b,n}$ because ℓ is a diffeomorphism. The noncrossing matchings are in bijection with the components of the Springer variety so $\bigcup_{a \in B^{n/2}} C_{a,n}$ is isomorphic to the $(n/2, n/2)$ Springer variety in the vector space $\ker X^{n/2}$. \square

The previous two lemmas allow us to conclude:

Theorem 5.4. *The $(n/2, n/2)$ Springer variety X_n is diffeomorphic to $\bigcup_{a \in B^{n/2}} S'_{a,n}$.*

Finally we observe this is diffeomorphic to Khovanov's construction of the Springer variety.

Theorem 5.5. *The $(n/2, n/2)$ Springer variety X_n is diffeomorphic to $\bigcup_{a \in B^{n/2}} S_{a,n}$.*

Proof. We prove that $\bigcup_{a \in B^{n/2}} S'_{a,n}$ is diffeomorphic to $\bigcup_{a \in B^{n/2}} S_{a,n}$ via the antipodal map γ of

Section 3.1. If a is a noncrossing matching then each arc $(i, j) \in a$ has exactly one even and one odd endpoint by Lemma 3.4, so γ will be the identity on exactly one coordinate i, j and the antipodal map on the other. Hence if $(x_1, \dots, x_n) \in \bigcup_{a \in B^{n/2}} S_{a,n}$ then $\gamma(x_1, \dots, x_n) \in$

$\bigcup_{a \in B^{n/2}} S'_{a,n}$. The differentiable map γ is its own inverse and so is a diffeomorphism. \square

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